

**THE RIGOROUS ANALYTICITY-UNITARITY PROGRAM  
AND ITS SUCCESSES****André MARTIN**

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**ABSTRACT**

We show how the combination of analyticity properties derived from local field theory and the unitarity condition (in particular positivity) leads to non-trivial physical results, including the proof of the “Froissart bound” from first principles and the existence of absolute bounds on the pion-pion scattering amplitude.

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I would like to begin by wishing a very happy birthday to Wolfhart Zimmermann. I have chosen a topic which is close to the interests of Wolfhart and as you will see soon, in which Wolfhart has made a crucial contribution which makes all the work made in the “pre-quark” era still valid now.

My task would have been much easier if the scheduled first speaker of this conference, Harry Lehmann, had been present. Unfortunately he was ill, and, at the time of writing this talk, we know that he left us. As we shall see all through what follows, the contributions of Harry Lehmann to that domain are many and all of them are fundamental.

In 1954, Gell-Mann, Goldberger and Thirring [1] proved that dispersion relation, previously developed in optics could be established for Compton Scattering:  $\gamma P \rightarrow \gamma P$ , from the existence of local fields satisfying the causality property

$$[A(x), A(y)] = 0 \quad \text{for} \quad (x - y)^2 < 0 ,$$

i.e., spacelike. This made it possible to express the real part of the forward scattering amplitude as an integral over the imaginary part of the forward scattering amplitude, i.e., by the “optical theorem”, an integral over the total cross-section for Compton Scattering. At the same time a general formulation of quantum field theory incorporating causality giving in particular general expression for scattering amplitude was developed by Lehmann, Zimmermann and Symanzik (LSZ) in their pioneering paper (in german!) in *Nuovo Cimento* [2].

On this basis, dispersion relations were “proposed” for massive particles in the work of Goldberger on the pion-nucleon scattering amplitude [3]. Soon, his “heuristic proof” was turned into a real proof by various authors using the LSZ formalism [4]. One of these proofs is due again to Harry Lehmann!

Before going on, I would like to explain that if these results, even after the discovery that protons and pions are not elementary but made of quarks, are still valid, it is thanks to a fundamental contribution of Wolfart Zimmermann entitled “On the bound state problem in quantum field theory” [5], in which it is proved that to a bound state we can associate a local operator. This constitutes an excellent answer to sceptics like Volodia Gribov [6] or Klaus Hepp [7] (qui brûle ce qu’il a adoré!).

Now I believe that it is necessary to give some technical details, even if most of you know about it.

In 3+1 dimensions (3 space, 1 time) the scattering amplitude depends on two variables energy and angle. For a reaction  $A + B \rightarrow A + B$

$$E_{c.m.} = \sqrt{M_A^2 + k^2} + \sqrt{M_B^2 + k^2} , \tag{1}$$

$k$  being the centre-of-mass momentum. The angle is designated by  $\theta$ . There are alternative variables:

$$s = (E_{CM})^2 , \quad t = 2k^2(\cos \theta - 1) \tag{2}$$

(Notice that physical  $t$  is NEGATIVE).

We shall need later an auxiliary variable  $u$ , defined by

$$s + t + u = 2M_A^2 + 2M_B^2 \quad (3)$$

The Scattering amplitude (scalar case) can be written as a partial wave expansion, the convergence of which will be justified in a moment:

$$F(s, \cos \theta) = \frac{\sqrt{s}}{k} \sum (2\ell + 1) f_\ell(s) P_\ell(\cos \theta) \quad (4)$$

$f_\ell(s)$  is a partial wave amplitude.

The Absorptive part, which coincides for  $\cos \theta$  real (i.e., physical) with the imaginary part of  $F$ , is defined as

$$A_s(s, \cos \theta) = \frac{\sqrt{s}}{k} \sum (2\ell + 1) \operatorname{Im} f_\ell(s) P_\ell(\cos \theta) \quad (5)$$

The Unitarity condition, implies, with the normalization we have chosen

$$\operatorname{Im} f_\ell(s) \geq |f_\ell(s)|^2 \quad (6)$$

which has, as a consequence

$$\operatorname{Im} f_\ell(s) > 0, \quad |f_\ell| < 1. \quad (7)$$

The differential cross-section is given by

$$\frac{d\sigma}{d\Omega} = \frac{1}{s} |F|^2,$$

and the total cross-section is given by the “optical theorem”

$$\sigma_{total} = \frac{4\pi}{k\sqrt{s}} A_s(s, \cos \theta = 1). \quad (8)$$

With these definitions, a dispersion relation can be written as:

$$F(s, t, u) = \frac{1}{\pi} \int \frac{A_s(s', t) ds'}{s' - s} + \frac{1}{\pi} \int \frac{A_u(u', t) du'}{u' - u} \quad (9)$$

with possible subtractions, i.e., for instance the replacement of  $1/(s' - s)$  by  $s^N/s'^N(s' - s)$  and the addition of a polynomial in  $s$ , with coefficients depending on  $t$ .

The scattering amplitude in the  $s$  channel  $A + B \rightarrow A + B$  is the boundary value of  $F$  for  $s + i\epsilon$ ,  $\epsilon > 0 \rightarrow 0$ ,  $s > (M_A + M_B)^2$ . In the same way the amplitude for  $A + \bar{B} \rightarrow A + \bar{B}$ ,  $\bar{B}$  being the antiparticle of  $B$  is given by the boundary value of  $F$  for  $u + i\epsilon$ ,  $\epsilon \rightarrow 0$ ,  $u > (M_A + M_B)^2$ . Here we understand the need for the auxiliary variable  $u$ .

The dispersion relation implies that, for fixed  $t$  the scattering amplitude can be continued in the  $s$  complex plane with two cuts. The scattering amplitude possesses the reality property,

i.e., for  $t$  real it is real between the cuts and takes complex conjugate values above and below the cuts.

In the most favourable cases, dispersion relations have been established for  $-T < t \leq 0$   $T > 0$ . A list of these cases have been given in 1958 by Goldberger [8] and has not been enlarged since then. It is given in the Table.

In the general case, even if dispersion relations are not proved, the crossing property of Bros, Epstein and Glaser states that the scattering amplitude is analytic in a twice cut plane, minus a finite region, for any negative  $t$  [9]. So it is possible to continue the amplitude directly from  $A + B \rightarrow A + B$  to the complex conjugate of  $A + \bar{B} \rightarrow A + \bar{B}$ . By a more subtle argument, using a path with fixed  $u$  and fixed  $s$  it is possible to continue directly from  $A + B \rightarrow A + B$  to  $A + \bar{B} \rightarrow A + \bar{B}$

At this point, we see already that one cannot dissociate analyticity, i.e., dispersion relations, and unitarity, since the discontinuity in the dispersion relations is given by the absorptive part. In the simple case of  $t = 0$ , the absorptive part is given by the total cross-section and the forward amplitude is given, as we said already for the case of Compton Scattering, by an integral over physical quantities.

It was recognized very early that the combination of analyticity and unitarity might lead to very interesting consequences and might give some hope to fulfill at least partially the  $S$  matrix Heisenberg program. This was very clearly stated already in 1956 by Murray Gell-Mann [10] at the Rochester conference. Later this idea was taken over by many people, in particular by Geff Chew. To make this program as successful as possible it seemed necessary to have an analyticity domain as large as possible. Dispersion relations are fixed  $t$  analyticity properties, in the other variable  $s$ , or  $u$  as one likes.

Another property derived from local field theory was the existence of the Lehmann ellipse [11], which states that for fixed  $s$ , physical, the scattering amplitude is analytic in  $\cos\theta$  in an ellipse with foci at  $\cos\theta = \pm 1$ .  $\cos\theta = 1$  corresponds to  $t = 0$  the ellipse therefore contains a circle

$$|t| < T_1(s) \tag{10}$$

In the Lehmann derivation  $T_1(s) \rightarrow 0$  for  $s \rightarrow (M_A + M_B)^2$  and  $s \rightarrow \infty$ .

The absorptive part is analytic in the larger ellipse, the “large” Lehmann ellipse, containing the circle

$$|t| < T_2(s) \tag{11}$$

with  $T_2(s) \rightarrow c > 0$  for  $s \rightarrow (M_A + M_B)^2$ ,  $T_2(s) \rightarrow 0$  for  $s \rightarrow \infty$ .

It was thought by Mandelstam that these two analyticity properties, dispersion relations and Lehmann ellipses, were insufficient to carry very far the analyticity-unitarity program. he proposed the Mandelstam representation [12] which can be written schematically as

# DISPERSION RELATIONS

## a) Proved relations

Process	Limitation in invariant momentum transfer	Continuation of absorptive part into the unphysical region by convergent partial wave expansion
$k + p \rightarrow k' + p'$		
$\pi + N \rightarrow \pi + N$	$T_{\max} = \frac{8m_\pi^2}{3} \quad \frac{2m_p+m_\pi}{2m_p-m_\pi}$	$0 \leq T < T_{\max}$
$\pi + \pi \rightarrow \pi + \pi$	$T_{\max} = 7m_\pi^2$ (now $28m_\pi^2$ )	$0 \leq T < T_{\max}$
$\gamma + N \rightarrow \gamma + N^{(*)}$	$T_{\max} = \mu^2 \left\{ \frac{(2m_p+m_\pi)^2}{4(m_p+m_\pi)^2} + \frac{2m_p+m_\pi}{m_p} \right\}$	$0 \leq T < T_{\max}$
$\gamma + N \rightarrow \pi + N^{(*)}$	$T_{\max} = F(0)^{(**)} \sim 3m_\pi^2$	$T_{th} \leq T < T_{\max}$
$e + N \rightarrow e + \pi + N^{(*)}$	$T_{\max} = F^{(**)}(\gamma); \quad \gamma \equiv k_0^2 - k^2$  $F(-9m_\pi^2) \sim 6m_\pi^2$	$T_{th} = \frac{m_p}{m_p+m_\pi} \times \frac{m_\pi-\gamma}{4}$

## b) Some unproved relations

	Mass restrictions appearing in proof based upon causality and spectrum; $T = 0$	Perturbation theory (every finite order)
$N + N \rightarrow N + N$	$m_\pi > (\sqrt{2} - 1)m_p$	proved for $T < \frac{m_\pi^2}{4}$
$K + N \rightarrow K + N$	complicated; not fulfilled by narrow margin	
$\pi + D \rightarrow \pi + D$	$\epsilon > \frac{m_\pi}{3} ; \quad m_D = 2m_p - \epsilon$	

$$\begin{aligned}
F = & \frac{1}{\pi^2} \int \frac{\rho(s', t') ds' dt'}{(s' - s)(t' - t)} \\
& + \text{circular permutations in } s, t, u \\
& + \text{one dimensional dispersion integrals} \\
& + \text{subtractions}
\end{aligned} \tag{12}$$

This representation is nice. It gives back the ordinary dispersion relations and the Lehmann ellipse when one variable is fixed, but it was never proved nor disproved for all mass cases, even in perturbation theory,. One contributor, Jean Lascoux, refused to co-sign a “proof”, which, in the end, turned out to be imperfect.

One very impressive consequence of Mandelstam representation was the proof, by Marcel Froissart, that the total cross-section cannot increase faster than  $(\log s)^2$ , the so-called “Froissart Bound” [13].

My own way to obtain the Froissart bound [14] was to use the fact that the Mandelstam representation implies the existence of an ellipse of analyticity in  $\cos \theta$  qualitatively larger than the Lehmann ellipse, i.e., such that it contains a circle  $|t| < R$ ,  $R$  fixed, independent of the energy. This has a consequence that  $\text{Im } f_\ell(s)$  decreases with  $\ell$  at a certain exponential rate because of the convergence of the Legendre polynomial expansion and of the polynomial boundedness, but on the other hand the  $\text{Im } f_\ell(s)$ ’s are bounded by unity because of unitarity [Eq. (7)]. taking the best bound for each  $\ell$  gives the Froissart bound.

To prove the Froissart bound without using the Mandelstam representation one must find a way to enlarge the “small” and the “large” Lehmann ellipses. In the autumn of 1965, I had very stimulating discussions with Harry Lehmann at the “Institut des Hautes Etudes Scientifiques” about an attempt made in this direction by Nakanishi in which he combined in a not very consistent way positivity and some analyticity properties derived from perturbation theory. He was using a domain shrinking to zero when the energy became physical and this lead nowhere. Finally, in December 1965 [15], I found the way out. The positivity of  $\text{Im } f_\ell$  implies, by using expansion (5),

$$\left| \left( \frac{d}{dt} \right)^n A_S(s, t) \right|_{-4k^2 \leq t \leq 0} \leq \left| \left( \frac{d}{dt} \right)^n A_S(s, t) \right|_{t=0} \tag{13}$$

To calculate

$$F(s, t) = \frac{1}{\pi} \int_{s_0} \frac{A_s(s' t) ds'}{s' - s}$$

(forget the left-hand cut and subtractions!), for  $s$  real  $< s_0$  one can expand  $F(s, t)$  around  $t = 0$ . From the property (13) one can prove that the successive derivatives can be obtained by differentiating under the integral. When one resums the series one discovers that this can be done not only for  $s$  real  $< s_0$ , but for any  $s$  and that the expansion has a domain of convergence in  $t$  independent of  $s$ . This means that the large Lehmann ellipse must contain a circle  $|t| < R$ . This is exactly what is needed to get the Froissart bound. In fact, in favourable cases,  $R = 4m_\pi^2$ ,

$m_\pi$  being the pion mass. A recipe to get a lower bound for  $R$  was found by Sommer [16]

$$R \leq \sup_{s_0 < s < \infty} T_1(s) \quad (14)$$

It was already known that for  $|t| < 4m_\pi^2$  the number of subtractions in the dispersion relations was at most two [17], and it leads to the more accurate bound [18]

$$\sigma_T < \frac{\pi}{m_\pi^2} (\log s)^2 \quad (15)$$

Notice that this is only a bound, not an asymptotic estimate.

In spite of many efforts the Froissart bound was never qualitatively improved, and it was shown by Kupsch [19] that if one uses only  $\text{Im } f_\ell \geq |f_\ell|^2$  and full crossing symmetry one cannot do better than Froissart.

Before 1972, rising cross-sections were a pure curiosity. Almost everybody believed that the proton-proton cross-section was approaching 40 millibarns at infinite energy. Only Cheng and Wu [20] had a QED inspired model in which cross-sections were rising and behaving like  $(\log s)^2$  at extremely high energy. Yet, Khuri and Kinoshita [21] took seriously very early the possibility that cross-sections rise and proved, in particular, that if the scattering amplitude is dominantly crossing even, and if  $\sigma_t \sim (\log s)^2$  then

$$\rho = \frac{\text{Re}F}{\text{Im}F} \sim \frac{\pi}{\log s} ,$$

where  $\text{Re}F$  and  $\text{Im}F$  are the real and imaginary part of the forward scattering amplitude.

In 1972, it was discovered at the ISR, at CERN, that the  $p - p$  cross-section was rising by 3 millibarns from 30 GeV c.m. energy to 60 GeV c.m. energy [22]. I suggested to the experimentalists that they should measure  $\rho$  and test the Khuri-Kinoshita predictions. They did it [23] and this kind of combined measurements of  $\sigma_T$  and  $\text{Re}F$  are still going on. In  $\sigma_T$  we have now more than a 50 % increase with respect to low energy values. For an up to date review I refer to the article of Matthiae [24]. It is my strong conviction that this activity should be continued with the future LHC. A breakdown of dispersion relation might be a sign of new physics due to the presence of extra compact dimensions of space according to N.N. Khuri [25]. Future experiments, especially for  $\rho$ , will be difficult because of the necessity to go to very small angles, but not impossible [26].

Before leaving the domain of high-energy scattering I would like to indicate the new version of the Pomeranchuk theorem. When it was believed that cross-sections were approaching finite limits, the Pomeranchuk theorem [27] stated that, under a certain assumption on the real part

$$\sigma_T(AB) - \sigma_T(A\bar{B}) \rightarrow 0$$

If cross-sections are rising to infinity, one can actually prove, according to Eden [28] and Kinoshita [29] that

$$\sigma_T(AB)/\sigma_T(A\bar{B}) \rightarrow 1 .$$

Now I would like to turn to another aspect of analyticity-unitarity. A consequence of the enlargement of the Lehmann ellipse is that, in the special case of  $\pi\pi \rightarrow \pi\pi$  scattering, one can, by using crossing symmetry, obtain a very large analyticity domain [30], but one can prove that the domain is smaller than the Mandelstam domain [31]. By playing with crossing symmetry and unitarity in a clever way (with years enormous progress has been made according to the Figure), one gets a bound on the scattering amplitude at the “symmetry point” which is [32]

$$\left| F(s = t = u = 4m_\pi^2/3) \right| < 4 ,$$

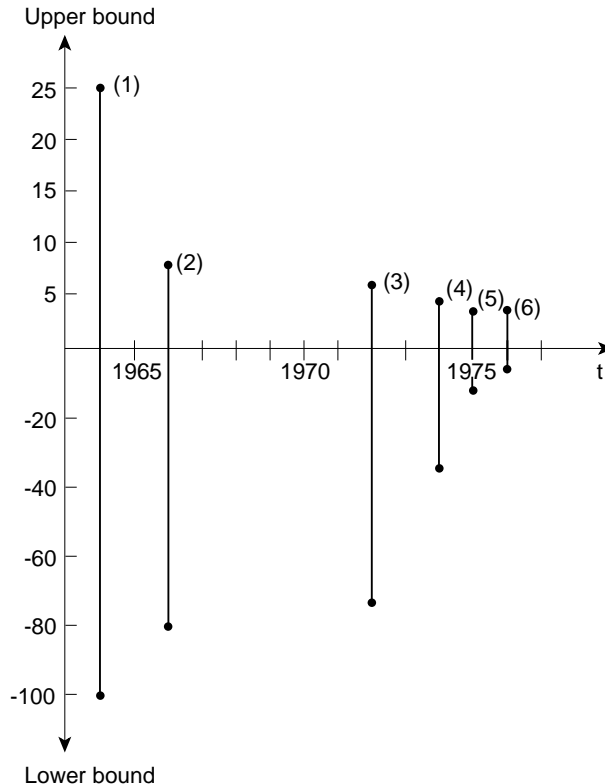


Figure 1: Bounds on the scattering amplitude at the symmetry point  $s = t = u = 4/3m_\pi^2$  as a function of time. Normalization:  $F(s = 4m_\pi^2, 0, 0) = \text{scattering length}$ .

where  $F$  is normalized in such a way that  $F(s = u, t = 0, u = 0)$  is the  $\pi_0\pi_0$  scattering length,  $a_{00}$ . One can also obtain a lower bound on the scattering length, the bound value being [33]

$$a_{00} > -1.75 \ (m_\pi)^{-1} ,$$

a number which is off the model predictions only by a factor 10.

Though these latter results may seem “useless”, they are remarkable, since they prove that the combination of analyticity and unitarity have a dynamical content.



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